Complete First-Order Axiomatization of Finite or Infinite M-extended Trees

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Abstract. We present in this paper an axiomatization of the structure of finite or infinite M-extended trees. This structure is an intuitive combination of the structure of finite or infinite trees with another structure M and expresses semantically an extension to trees of the model M. Having a structure $M = (D_M, F_M, R_M)$, we define the structure of finite or infinite M-extended tree $Ext_M = (D, F, R)$ whose domain D consists of trees labelled by elements of $D_M \cup F$, where F is an infinite set of function symbols containing F_M and another infinite set of function symbols disjoint from F_M. For each n-ary function symbol $f \in F$, the operation $f(a_1, .., a_n)$ is evaluated in M and produces an element of D_M if $f \in F_M$ and all the a_i are elements of D_M, or is a tree whose root is labelled by f and whose immediate children are $a_1, .., a_n$ otherwise. The set of relations R contains R_M and another relation which distinguishes the elements of D_M from the others. Using a first-order axiomatization T of M, we give a first-order axiomatization $\mathcal{T}$ of the structure Ext_M and show that if T is *flexible* then $\mathcal{T}$ is *complete*. The flexible theories are particular theories whose function and relation symbols have some elegant properties which enable us to handle formulae more easily.

1 Introduction

Recall that a tree built on a set E is essentially a hierarchized set of nodes labelled by the elements of E. To each element e of E corresponds an operation f, called *construction operation*, which, starting from a sequence $a_1, \dots, a_n$ of trees, builds the tree whose top node is labelled e and whose sequence of immediate children is $a_1, \dots, a_n$.

The algebra of finite or infinite trees plays a fundamental act in computer science: it is a model for composed data known as *record* in Pascal or *structure* in C. The construction operation corresponds to the creation of a new record, i.e. of a cell containing an elementary information possibly followed by n cells, each one pointing to a record. Circuit of pointers correspond to infinite trees.

As early as 1976, G. Huet proposed an algorithm for unifying infinite terms, that is solving equations in that algebra [11]. B. Courcelle has studied the properties of infinite trees in the scope of recursive program schemes [6]. A. Colmerauer has described the execution of Prolog II, III and IV programs in terms of solving equations and disequations in that algebra [4, 3, 1]. The unification of finite terms, i.e. solving conjunctions of equations in the theory of finite trees has first been studied by A. Robinson [18]. Some better algorithms with better complexities has been proposed after by M.S. Paterson and M.N.Wegman [16] and A. Martelli and U. Montanari [15]. Solving conjunctions of equations in the theory of infinite trees has been studied by G. Huet [11], by A. Colmerauer [4] and by J. Jaffar [12]. Solving conjunctions of equations and disequations in the theory of possibly infinite trees has been studied by A. Colmerauer [4] and H.J. Bürckert [2]. An incremental algorithm for solving conjunctions of equations and disequations on rational trees has been proposed after by V.Ramachandran and P. Van Hentenryck [17]. On the other hand, there exists a quantification elimination algorithm which transforms a first-order formula into a boolean combination of simple ones. In the case of infinite trees with a finite set of function symbols we can refer to the work of M.J. Maher [14] and H. Comon [5]. M.J. Maher has summarized all these cases and proposed a complete axiomatizations for different sets of trees equipped with construction operations [14].

In this paper, we give and justify an axiomatization of the structure of finite or infinite M-extended trees. This structure is an intuitive combination of the structure of trees with another structure M and can be seen semantically as an extension to trees of the model M. Having a structure $M = (D_M, F_M, R_M)$ together with its domain D_M, its set of operations F_M and its set of relations R_M, we define the M-extended tree structure $Ext_M = (D, F, R)$ whose domain D consists of trees labelled by elements of $D_M \cup F$, where F is an infinite set of function symbols containing F_M and another infinite set of function symbols disjoint from F_M. For each n-ary function $f \in F$, the operation $f(a_1, ..., a_n)$ is evaluated in M and produces an element of D_M if $f \in F_M$ and all the a_i are elements of D_M, or is a tree whose root is labelled by f and whose immediate children are $a_1, ..., a_n$ otherwise. The set of relations R is built essentialy from R_M. In the case where M is the set of rational numbers together with the operations of addition and substraction and a linear dense order relation we can refer to Prolog III and IV whose execution has been modelized by A. Colmerauer [4, 1] using this M-extended trees.

The paper is organized in four sections followed by a conclusion. This introduction is the first section. In the second section we recall the Maher's structure of finite or infinite trees and introduce the M-extended trees structure for any model M. In the third section, we present our general sufficient conditions for the completeness of any first-order theory. Then, having a first-order axiomatization T of M, we give a first-order axiomatization $\mathcal{T}$ of finite or infinite M-extended trees. Finally we present in the fourth section a new class of theories that we call *flexible* and show that if T is flexible then $\mathcal{T}$ is complete. To show the completeness of $\mathcal{T}$ for any flexible theory T we use the general sufficient conditions presented in the third section. The definition of the M-extended trees, the axiomatization of $\mathcal{T}$, the definition of flexible theories and the proof of the completeness of $\mathcal{T}$ for every flexible theory T are our main contribution in this paper.

2 Extension to trees of a model M

2.1 Finite or infinite trees

Let F be an infinite set of *function symbols* and R be a set of *relation symbols.* To each element of $F \cup R$ is associated an integer, its *arity.* The arities are non-negative for elements of F and are positive for elements of R. An n-ary symbol is a symbol with arity n. A *constant* is a 0-ary symbol.

Let N be a set of words of positive integers, including the empty word ϵ. Let "." denote concatenation of word. A *tree built on* F is a mapping $a : E \to F$, for some non-empty subset E of N such that each element $i_1 \dots i_k$ (with $k \geq 0$) satisfies two conditions: (1) if $k > 0$ then $i_1 \dots i_{k-1} \in E$ and (2) if $a(i_1 \dots i_k) = f$ and f has arity n, then $i_1 \dots i_k i_{k+1} \in E$ if and only if $1 \leq i_{k+1} \leq n$.

The subtree of the tree a at $n \in E$ is the mapping $a' : E' \to F$ where $D' = \{d | n.d \in E\}$ and $a'(d) = a(n.d)$.

The set of all trees built on F is denoted A. To each n-ary function symbol f we associate a function from A^n to A also denoted f such that $f(a_1, \dots, a_n) = a$ where $a(\epsilon) = f$ and $a(i.d) = a_i(d)$ for $1 \leq i \leq n$ and d a node. These functions are called construction operations. The set of trees A with these construction operations forms the tree structure or tree algebra.

2.2 Finite or infinite M-extended tree structure

We are given now once for all a structure $M = (D_M, F_M, R_M)$ with its domain D_M, its set of functions F_M and its set of relations R_M. Let F be an infinite set of function symbols containing the set F_M and another infinite set of function symbols disjoint from F_M. Let R be the set of relation symbols $R_M \cup \{p\}$, with p a unary relation symbols which does not belong to R_M. The extension to trees of the model M, quite simply called M-extended trees model is the model $Ext_M = (D, F, R)$ defined as follows:

- the domain D is the set of the trees built on $F \cup D_M$ where each element $f \in F$ of arity n is considered as a label of arity n and each element of D_M is considered as a label of arity 0,
- to each n-ary element f of F is associated a function $f : D^n \to D$ such that $f(a_1, .., a_n)$ is the result of f on $(a_1, .., a_n)$ in D_M, if $f \in F_M$ and $a_i \in D_M$ for all i, and is the result of the construction operation f on $(a_1, .., a_n)$ otherwise,
- to each n-ary relation symbols r of $R - \{p\}$ is associated the set $r^{Ext_M} = r^M$. To the unary relation symbols p is associated the set $p^{Ext_M} = D_M$.

3 Theory of finite or infinite M-extended trees

Let V an infinite countable set of *variables.* A *term* is an expression of the form x or $ft_1 \dots t_n$ where $n \geq 0$, f an n-ary symbol in F and the t_i's are shorter terms.

A M*-term* is either a variable or a term whose function symbols are elements of F_M. A *formula* is an expression of the forms

$$s{=}t,\ rt_1..t_n,\ \mathit{true},\ \mathit{false},\ \neg(\varphi),\ (\varphi\wedge\psi),\ (\varphi\vee\psi),\ (\varphi\rightarrow\psi),\ (\varphi\leftrightarrow\psi),\ \exists x\varphi,\ \forall x\varphi,$$

where $x \in V$, s, t and the t_i's are terms, r is an n-ary relation symbol in R and φ and ψ are shorter formulae. Formulae of the first form are called *equations* and of the second form *relations.* A M*-equation* is an equation of M-terms and a M*-relation* is a relation $rt_1...t_n$ with $r \in R_M$ and the t_i's M-terms.

An occurrence of a variable x in a formula is *bound* if it occurs in a sub-formula of the form $(\exists x\varphi)$ or $(\forall x\varphi)$. It is *free* otherwise. The *free variables* of a formula are those which have at least a free occurrence in the formula. For each formula φ, we denote by $\mathit{var}(\varphi)$ the set of all free variables of φ.

We call *instantiation* of a formula φ by individuals of D_M the obtained formula from φ in which for each free variable x in φ, we replace each free occurrence of x by the same individual i of D_M.

3.1 Theory and complete theory

Let $\bar{x} = x_1 \dots x_n$ and $\bar{y} = y_1 \dots y_n$ be two *vectors of variables* of the same length. Let ψ, ϕ, φ and $\varphi(\bar{x})$ be formulae. We write

$$\begin{array}{ll} \exists\bar{x}\,\varphi & \text{for } \exists x_1...\exists x_n\,\varphi, \\ \forall\bar{x}\,\varphi & \text{for } \forall x_1...\forall x_n\,\varphi, \\ \exists?\bar{x}\,\varphi(\bar{x}) & \text{for } \forall\bar{x}\forall\bar{y}\,\varphi(\bar{x})\wedge\varphi(\bar{y}) \rightarrow \bigwedge_{i\in\{1,\dots,n\}} x_i = y_i, \\ \exists!\bar{x}\,\varphi & \text{for } (\exists\bar{x}\,\varphi)\wedge(\exists?\bar{x}\,\varphi). \end{array}$$

Note that the formulae $\exists?\varepsilon\varphi$ and $\exists!\varepsilon\varphi$ are respectively equivalent to *true* and to φ in any model M. Theses quantifiers are just convenient notations and can be expressed in the first-order level.

Definition 3.1.1 *Let $\Psi(u)$ be a set of formulas having at most u as a free variable. We write $M \models \exists_{o\ \infty}^{\Psi(u)} x\,\varphi(x)$, iff for any instantiation $\exists x\,\varphi'(x)$ of $\exists x\,\varphi(x)$ by individuals of D_M one of the following properties holds:*

- *the set of the individuals i of D_M such that $M \models \varphi'(i)$, is empty,*
- *for all finite sub-set $\{\psi_1(u), .., \psi_n(u)\}$ of elements of $\Psi(u)$, the set of the individuals i of D_M such that $M \models \varphi'(i) \wedge \bigwedge_{j\in\{1,\dots,n\}} \neg\psi_j(i)$ is infinite.*

A *theory* is a set of propositions. We say that the model M is a *model of* T iff for each element φ of T, $M \models \varphi$. If φ is a formula, we write $T \models \varphi$ iff for each model M of T, $M \models \varphi$. A theory T is *complete* if for each proposition φ, either $T \models \varphi$ or $T \models \neg\varphi$. A *complete axiomatization* of a structure M is a recursive set T of propositions such that for each proposition φ, $T \models \varphi$ iff $M \models \varphi$.

In what follows we use the abbreviation wnfv for *"without new free variables"*. By saying a formula φ is equivalent to a wnfv formula ψ in T we mean $T \models \varphi \leftrightarrow \psi$ and ψ does not contain other free variables than those of φ. The following theorem states general sufficient conditions for the completeness of a theory T.

Theorem 3.1.2 *[9, 10] A theory T is complete if there exists a set $\Psi(u)$ of formulas, having at most u as free variable, a set A of formulas, closed under conjunction and renaming, a set A' of formulas of the form $\exists\bar{x}\alpha$ with $\alpha \in A$, and a sub-set A'' of A such that:*

1. *every flat atomic formula is equivalent in T to a wnfv Boolean combination of basic formulas of the form $\exists\bar{x}\alpha$ with $\alpha \in A$,*
2. *every formula without free variables of the form $\exists\bar{x}'\alpha' \wedge \alpha''$ with $\exists\bar{x}'\alpha' \in A'$ and $\alpha'' \in A''$ is equivalent either to* false *or to* true *in T,*
3. *every formula of the form $\exists\bar{x}\,\alpha \wedge \psi$, with $\alpha \in A$ and ψ any formula, is equivalent in T to a wnfv formula of the form:*

$$\exists\bar{x}'\,\alpha' \wedge (\exists\bar{x}''\,\alpha'' \wedge (\exists\bar{x}'''\,\alpha''' \wedge \psi)),$$

   with $\exists\bar{x}'\,\alpha' \in A'$, $\alpha'' \in A''$, $\alpha''' \in A$ and $T \models \forall\bar{x}''\alpha'' \to \exists!\bar{x}'''\alpha'''$,
4. *if $\exists\bar{x}'\alpha' \in A'$ then $T \models \exists?\bar{x}'\,\alpha'$ and for each free variable y in $\exists\bar{x}'\alpha'$, at least one of the following properties holds:*
   - *$T \models \exists?y\bar{x}'\,\alpha'$,*
   - *there exists $\psi(u) \in \Psi(u)$ such that $T \models \forall y\,(\exists\bar{x}'\,\alpha') \to \psi(y)$,*
5. *if $\alpha'' \in A''$ then*
   - *the formula $\neg\alpha''$ is equivalent in T to a wnfv formula of the form $\bigvee_{i\in I}\alpha_i$ with $\alpha_i \in A$,*
   - *for each x'', the formula $\exists x''\alpha''$ is equivalent in T to a wnfv formula which belongs to A'',*
   - *for each x'', $T \models \exists_{o\,\infty}^{\Psi(u)} x''\,\alpha''$.*

3.2 Axiomatization of the structure of M-extended trees

M. Maher has introduced a complete axiomatization of the structure of finite or infinite trees built on an infinite set F [14]. The axiomatization is the set of propositions of the following forms:

$$\begin{array}{ll} 1 & \forall\bar{x}\forall\bar{y}\, f\bar{x} = f\bar{y} \to \bigwedge_i x_i = y_i, \\ 2 & \forall\bar{x}\forall\bar{y}\, \neg f\bar{x} = g\bar{y}, \\ 3 & \forall\bar{x}\exists!\bar{z}\, \bigwedge_i z_i = t_i(\bar{z},\bar{x}), \end{array}$$

where $f, g \in F$, x, y, z are variables, $\bar{x}$ is vector of variables x_i, $\bar{y}$ is vector of variables y_i, $\bar{z}$ is vector of distinct variables z_i and where $t_i(\bar{x},\bar{z})$ is a term which begins by an element of F followed by variables taken from $\bar{x}$ or $\bar{z}$.

The first axiom is called *axiom of explosion*, the second *axiom of conflict of symbols* and the third *axiom of unique solution*.

Let T be an axiomatization of the structure $M = (D_M, F_M, R_M)$. Using this axiomatization, let us now define an axiomatization $\mathcal{T}$ of the structure of finite or infinite M-extended trees together with the sets F and R (defined in section 2.2) as function and relation symbols.

Definition 3.2.1 An axiomatization $\mathcal{T}$ of the structure of finite or infinite M-extended trees is the set of propositions of the following forms where $\bar{x}$, $\bar{y}$ are vectors of variables x_i, y_i.

1. explosion: for each $f \in F$

$$\forall \bar{x} \forall \bar{y} \; \neg pf\bar{x} \wedge \neg pf\bar{y} \wedge f\bar{x} = f\bar{y} \rightarrow \bigwedge_i x_i = y_i$$

2. conflict of symbols: for f and g distinct symbols in F

$$\forall \bar{x} \forall \bar{y} \; f\bar{x} = g\bar{y} \rightarrow pf\bar{x} \wedge pg\bar{y}$$

3. unique solution

$$\forall \bar{x} \forall \bar{y} \; (\bigwedge_i px_i) \wedge (\bigwedge_j \neg py_j) \rightarrow \exists! \bar{z} \bigwedge_k (pz_i \wedge z_k = t_k(\bar{x}, \bar{y}, \bar{z}))$$

   where $\bar{z}$ is a vector of distinct variables z_i, $t_k(\bar{x}, \bar{y}, \bar{z})$ is a term expressed by a function symbol f_k followed by variables taken from $\bar{x}, \bar{y}, \bar{z}$, moreover, if $f_k \in F_M$, the term $t_k(\bar{x}, \bar{y}, \bar{z})$ contains at least one variable from $\bar{y}$ or $\bar{z}$
4. relations of R_M: for each $r \in R_M$,

$$\forall \bar{x} \; r\bar{x} \rightarrow \bigwedge_i px_i$$

5. operations of F_M: for each $f \in F_M$,

$$\forall \bar{x} \; pf\bar{x} \leftrightarrow \bigwedge_i px_i$$

   (this axiom, in the case of f a constant in F_M, becomes pf)
6. elements not in M: for each $f \in F - F_M$,

$$\forall \bar{x} \; \neg pf\bar{x}$$

7. existence of at least one element satisfying p (only if F_M does not contains 0-arity function symbols):

$$\exists x \, px,$$

8. the extension of axioms of T: all axioms obtained by the following transformation of an axiom φ of T : While it is possible replace all sub-formula of φ which is of the form $\exists \bar{x} \, \psi$, but not of the form $\exists \bar{x} \, (\bigwedge px_i) \wedge \psi'$, by $\exists \bar{x} \, (\bigwedge px_i) \wedge \psi$ and all sub-formula of φ which is of the form $\forall \bar{x} \, \psi$, but not of the form $\forall \bar{x} \, (\bigwedge px_i) \rightarrow \psi'$, by $\forall \bar{x} \, (\bigwedge px_i) \rightarrow \psi$.

Example 3.2.2 Let M be the structure of the rational numbers together with the operations of addition, substraction and a linear dense order relation without endpoints. In this case D_M is the set of the rational numbers, $F_M = \{+, -, 0, 1\}$ and $R_M = \{<\}$. Let a be a positive integer and let $t_1, ..., t_n$ be terms. Let us denote by:

- $t_1 < t_2$, the term $< t_1 t_2$,
- $t_1 + t_2$, the term $+ t_1 t_2$,
- $t_1 + t_2 + t_3$, the term $+ t_1 (+ t_2 t_3)$,
- $-a t_1$, the term $\underbrace{(-t_1) + \cdots + (-t_1)}_{a}$,
- $0 t_1$, the term 0,
- $a t_1$, the term $\underbrace{t_1 + \cdots + t_1}_{a}$,
- a the term $\underbrace{1 + \cdots + 1}_{a}$.

The axiomatization T of the structure M is of the form

1 $\forall x \forall y \, x + y = y + x$,
2 $\forall x \forall y \forall z \, x + (y + z) = (x + y) + z$,
3 $\forall x \, x + 0 = x$,
4 $\forall x \, x + (-x) = 0$,
5_n $\forall x \, nx = 0 \rightarrow x = 0$,
6_n $\forall x \, \exists! y ny = x$, $\quad (n \neq 0)$
7 $\forall x \, \neg x < x$,
8 $\forall x \forall y \forall z \, (x < y \wedge y < z) \rightarrow x < z$,
9 $\forall x \forall y \, (x < y \vee x = y \vee y < x)$,
10 $\forall x \forall y \, x < y \rightarrow (\exists z \, x < z \wedge z < y)$,
11 $\forall x \, \exists y \, x < y$,
12 $\forall x \, \exists y \, y < x$,
13 $\forall x \, \forall y \, \forall z \, x < y \rightarrow (x + z < y + z)$,
14 $0 < 1$.

Using the transformations of Definition 3.2.1, the axiomatization $\mathcal{T}$ of the M-extended trees theory is of the form:

1 $\forall \bar{x} \forall \bar{y} \, ((\neg p \, f \bar{x}) \wedge (\neg p \, f \bar{y}) \wedge f \bar{x} = f \bar{y}) \rightarrow \bigwedge_i x_i = y_i$,
2 $\forall \bar{x} \forall \bar{y} \, f \bar{x} = g \bar{y} \rightarrow p \, f \bar{x} \wedge p \, g \bar{y}$,
3 $\forall \bar{x} \forall \bar{y} \, (\bigwedge_{i \in I} p \, x_i) \wedge (\bigwedge_{j \in J} \neg p \, y_j) \rightarrow (\exists! \bar{z} \, \bigwedge_{k \in K} (\neg p \, z_k \wedge z_k = t_k(\bar{x}, \bar{y}, \bar{z})))$,
4 $p0$,
5 $p1$,
6 $\forall x \forall y \, x < y \rightarrow (p \, x \wedge p \, y)$,
7 $\forall x \forall y \, p \, x + y \leftrightarrow p \, x \wedge p \, y$,
8 $\forall x \, p \, - x \leftrightarrow p \, x$,
9 $\forall \bar{x} \, \neg p \, h \bar{x}$,
10 $\forall x \forall y \, (p \, x \wedge p \, y) \rightarrow x + y = y + x$,
11 $\forall x \forall y \forall z \, (p \, x \wedge p \, y \wedge p \, z) \rightarrow x + (y + z) = (x + y) + z$,
12 $\forall x \, p \, x \rightarrow x + 0 = x$,
13 $\forall x \, p \, x \rightarrow x + (-x) = 0$,
14_n $\forall x \, p \, x \rightarrow (nx = 0 \rightarrow x = 0)$,
15_n $\forall x \, p \, x \rightarrow \exists! y p \, y \wedge ny = x$, $\quad (n \neq 0)$
16 $\forall x \, p \, x \rightarrow \neg x < x$,
17 $\forall x \forall y \forall z \, p \, x \wedge p \, y \wedge p \, z \rightarrow ((x < y \wedge y < z) \rightarrow x < z)$,
18 $\forall x \forall y \, (p \, x \wedge p \, y) \rightarrow (x < y \vee x = y \vee y < x)$,
19 $\forall x \forall y \, (p \, x \wedge p \, y) \rightarrow (x < y \rightarrow (\exists z \, p \, z \wedge x < z \wedge z < y))$,
20 $\forall x \, p \, x \rightarrow (\exists y \, p \, y \wedge x < y)$,
21 $\forall x \, p \, x \rightarrow (\exists y \, p \, y \wedge y < x)$,
22 $\forall x \, \forall y \, \forall z \, (p \, x \wedge p \, y \wedge \, p \, z) \rightarrow (x < y \rightarrow (x + z < y + z))$,
23 $0 < 1$,

where f and g are two distinct function symbols taken from F, $h \in F - F_M$, x, y, z are variables, $\bar{x}$ is a vector of variables x_i, $\bar{y}$ is a vector of variables y_i, $\bar{z}$ is vector of distinct variables z_i and where $t_k(\bar{x}, \bar{y}, \bar{z})$ is a term which begins by a function symbol f_k element of F followed by variables taken from $\bar{x}$ or $\bar{y}$ or $\bar{z}$, moreover, if $f_k \in F_M$ then $t_k(\bar{x}, \bar{y}, \bar{z})$ contains at least a variable taken from $\bar{y}$ or $\bar{z}$. This theory has been used by A. Colmerauer to modelize the execution of Prolog III and IV [4, 1].

4 Completeness of $\mathcal{T}$

We suppose that the variables of V are ordered by a strict linear dense order relation denoted by $\succ$. We call *leader* of an M-equation α the greatest variable x of all variables in α, according to the order $\succ$, such that $M \models \exists! x \alpha$.

4.1 Flexible structure

The model M is called *flexible* if for each conjunction α of M-equations and each conjunction β of M-relations:

1. α is equivalent in M either to *false* or to a wnfv conjunction α' of M-equations whose each element has a distinct leader which has one and only occurrence in α', and for all variable $x \in var(\alpha')$ we have $M \models \exists! x \alpha'$,
2. the formula $\neg\beta$ is equivalent in M to a wnfv disjunction of M-equations and M-relations,
3. for all $x \in V$
   - the formula $\exists x\, \beta$ is equivalent in M either to *false* or to a quantifier free conjunction of M-relations,
   - for all $x \in V$ and for all instantiation $\exists x\, \beta'(x)$ of $\exists x\, \beta(x)$ by individuals of D_M, either $M \models \neg\exists x \beta'(x)$ or there exists an infinite set of individuals i of D_M such that $M \models \beta'(i)$.

A theory T is called flexible iff all its models are flexible.

Property 4.1.1 *If T is flexible then it is complete.*

4.2 Blocks and solved blocks in $\mathcal{T}$

Definition 4.2.1 *A* block *is a conjunction α of formulae of the following forms:*

- *true, false, px, $\neg px$,*
- *$x = y$, $x = fx_1 \ldots x_n$, with $f \in F$,*
- *$t_1 = t_2 \wedge \bigwedge_{i=1}^{n} px_i$, where $\{x_1, \ldots, x_n\}$ is the set of variables which occur in the M-equation $t_1 = t_2$,*
- *$rt_1 \ldots t_n$, where $r \in R_M$ and the t_i's are M-terms,*

and such that α contains px or $\neg$px for each variable $x \in$ var(α). A relation block *is a block without equations. An* equation block *is a block without M-relations and where each variable has an occurrence in at least one equation.*

Definition 4.2.2 *If a block α has a sub-formula of the form*

$$x_0 = t_0(x_1) \wedge x_1 = t_1(x_2) \wedge \cdots \wedge x_{n-1} = t_{n-1}(x_n) \wedge \bigwedge_{i=0}^{n-1} \neg p x_i,$$

where x_{i+1} has an occurrence in the term $t_i(x_{i+1})$, then the variable x_n and the equation $x_{n-1} = t_{n-1}(x_n)$ are called reachable *from x_0 in α.*

Property 4.2.3 *Let α be a block. If all the variables of $\bar{x}$ are reachable in α from free variables of $\exists \bar{x} \alpha$, then $\mathcal{T} \models \exists ? \bar{x} \alpha$.*

Definition 4.2.4 A block α is called *well-typed* iff α does not contain sub-formulae of one of the following forms:

- $p\,x \wedge \neg p\,x$,
- $x = h\bar{y} \wedge p\,x$, with $h \in F - F_M$,
- $x = f_0 \wedge \neg p\,x$, with f_0 a constant of F_M,
- $x_0 = f x_1 ... x_n \wedge \neg p\,x_0 \wedge \bigwedge_{i=1}^{n} p\,x_i$, with $f \in F_M$,
- $x_0 = f x_1 ... x_n \wedge p\,x_0 \wedge \neg p\,x_i$, with $f \in F$
- $x_0 = x_1 \wedge p\,x_0 \wedge \neg p\,x_1$,
- $x_0 = x_1 \wedge \neg p\,x_0 \wedge p\,x_1$,
- $r t_1 ... t_n \wedge \neg p\,x_i$ with $r \in R_M$ and x_i a variable which occurs in the M-relation $r t_1 ... t_n$.

Definition 4.2.5 *Let t_1 be a term. Let t_2 and t_3 be two M-terms. Let α be a well-typed equation block. Either $x = t_1 \wedge \neg px$ is a sub-formula of α. In this case, x is called α-leader of the equation $x = t_1$. Else $t_2 = t_3 \wedge \bigwedge_{i \in \mathrm{var}(t_2 = t_3)} pi$ is a sub-formula of α. In this case, the greatest variable in $var(t_2 = t_3)$ according to the order $\succ$ such that $\mathcal{T} \models \exists ! x\, t_2 = t_3 \wedge \bigwedge_{i \in var(t_2 = t_3)} pi$ is called α-leader of the equation $t_2 = t_3$.*

Definition 4.2.6 *A block α is called* solved block*, iff:*

1. *α is well-typed and does not contain formulae of the form $t_1 = t_2$ or $r t_1 ... t_n$ with $r \in R_M$ and the t_i's terms which does not contain variables,*
2. *for each equation $x = y$ in α, $x \succ y$,*
3. *each equation in α has a distinct α-leader which does not occur in M-relations of α,*
4. *if px and py are sub-formulas of α with x and y two α-leaders of two equations α_1, α_2 of α then $x \notin \mathrm{var}(\alpha_2)$,*
5. *for all variable x which occurs in an equation of α we have $\mathcal{T} \models \exists ? x \alpha$.*

Property 4.2.7 *Let α be a solved equation block different from the formula* false *and let $\bar{x}$ be the set of the α-leaders of the equations of α. We have $\mathcal{T} \models \exists ! \bar{x} \alpha$.*

Property 4.2.8 *If T is flexible then each block is equivalent in $\mathcal{T}$ to a solved block.*

4.3 Completeness of $\mathcal{T}$

Theorem 4.3.1 *If T is a flexible theory then $\mathcal{T}$ is complete.*

We show this theorem using Theorem 3.1.2. The sets $\Psi(u)$, A, A' and A'' are chosen as follows:

- $\Psi(u)$ is the set of the formulae of the form $\exists \bar{y}\, u = f\bar{y} \wedge \neg p\, u$, with $f \in F$ not a constant.
- A is the set of blocks.
- A' is the set of the formulae of the form $\exists \bar{x}' \alpha'$, where:
  - all the variables of $\bar{x}'$ are reachable in α' from free variables of $\exists \bar{x}' \alpha'$,
  - α' is a solved equation block, different from the formula *false*, and where the order $\succ$ is such that all the variables of $\bar{x}'$ are greater than the free variables of $\exists \bar{x}' \alpha'$,
  - all the equations of α' of the form $x_0 = f x_1 ... x_n$ with $f \in F - F_M$ are reachable in α' from free variables of $\exists \bar{x}' \alpha'$,
  - if the M-equation $t_1 = t_2$ is a sub-formula of α' then, each variable x_i which occurs in it is either a free variable of $\exists \bar{x}' \alpha'$ or reachable in α' from free variables of $\exists \bar{x}' \alpha'$,
- A'' is the set of solved relation blocks.

5 Conclusion

We have defined in this paper the structure of the M-extended trees for any model M. This structure can be considered as a combination of the structure of finite or infinite trees with the structure M. Having used an axiomatization T of M we have given a first-order axiomatization $\mathcal{T}$ of the M-extended trees structure and have shown that if T is flexible then $\mathcal{T}$ is complete. To prove the completeness in this case, we have used our general sufficient condition. From this condition we can extract a general algorithm for solving first-order constraints in $\mathcal{T}$. Due to lack of space we cannot present this algorithm in this paper. Just note that this algorithm uses the block defined in our paper and transforms any formula φ in a particular formula ψ called *solved formula* equivalent to φ in $\mathcal{T}$. In particular if φ has no free variables then ψ is either the formula *true* or the formula *false*. The correctness of our algorithm is another proof of the completeness of $\mathcal{T}$ of each flexible theory T.

There exists a lot real and practical problems which can be represented by full first-order formulae on M-extended trees. We can site for example the works of A. Colmerauer [4, 1] who has realized the execution of Prolog III and IV using the M-extended trees where M is the structure of the rational numbers together with the operations of addition and substraction and linear dense order relation.

On the other hand S. Vorobyov [19] have shown that the problem of deciding if a proposition without free variables is true or not in the tree theory is non-elementary, i.e. the complexity of all algorithm which solve it is not bounded by a tower of powers of $2's$ (with a top down evaluation) with a fixed height. A.

Colmerauer and B. Dao [8, 7] have also given a proof of non-elementary complexity of solving constraints in the tree theory. Thus, it is normal that our sufficient condition is complex and the properties of our blocks uses some nonclassical quantifiers. Nevertheless we hope find some interesting class of complexities in the implementation of our algorithm as it has been done in [8] in the theory of finite or infinite trees.

Actually we try to show the completeness of $\mathcal{T}$ where M is the structure of the real numbers together with addition, substraction, multiplication and a linear dense order relation. We also study the complexity and the expressiveness of the first-order constraints in $\mathcal{T}$ as it has done in [7, 8].

Acknowledgements We thank Alain Colmerauer for our many discussions and his help in the organization and the drafting of this paper. We thank him too, for the definitions and proof of his course of DEA. We dedicate to him this paper with our best wishes for a speedy recovery.

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Appendix

Theorem 3.1.2 If T is a flexible theory then $\mathcal{T}$ is complete.

Proof. We show this theorem using Theorem 3.1.2. The sets $\Psi(u)$, A, A' and A'' are chosen as follows:

5.1 Choice of the sets $\Psi(u)$, A, A' and A''

- $\Psi(u)$ is the set of the formulae of the form $\exists \bar{y}\, u = f\bar{y} \wedge \neg p\, u$, with $f \in F - F_0$.
- A is the set of blocks.
- A' is the set of the formulae of the form $\exists \bar{x}' \alpha'$, where:
  - all the variables of $\bar{x}'$ are reachable in α' from free variables of $\exists \bar{x}' \alpha'$,
  - α' is a solved equation block, different from the formula *false*, and where the order $\succ$ is such that all the variables of $\bar{x}'$ are greater than the free variables of $\exists \bar{x}' \alpha'$,
  - all the equations of α' of the form $x_0 = f x_1 ... x_n$ with $f \in F - F_M$ are reachable in α' from free variables of $\exists \bar{x}' \alpha'$,
  - if the M-equation $t_1 = t_2$ is a sub-formula of α' then, each variable x_i which occurs in it is either a free variable of $\exists \bar{x}' \alpha'$ or reachable in α' from free variables of $\exists \bar{x}' \alpha'$,
- A'' is the set of solved relation blocks.

Note 5.1.1 *Note that, A is closed under conjunction and renaming and A'' is a sub-set of A.*

Let T be a flexible theory. Let us show that $\mathcal{T}$ satisfies the five conditions of Theorem 3.1.2.

5.2 $\mathcal{T}$ satisfies the first condition

Let us show that every flat atomic formula φ is equivalent in $\mathcal{T}$ to a wnfv Boolean combination of basic formulas, i.e. a wnfv Boolean combination of formulas of the form $\exists \bar{x}\alpha$ with $\alpha \in A$. Let φ be a flat atomic formula. If φ is of the form *true*, *false*, px or $\neg p\, x$, then φ is a block which is quantified by the empty vector ε, thus, φ is equivalent in $\mathcal{T}$ to $\exists \varepsilon \varphi$ with $\varphi \in A$, which is clearly a wnfv Boolean combination of basic formulas. Else, the following equivalences, after having distributed the $\wedge$ on the $\vee$ and the $\exists$ on the $\vee$, give the adequate Boolean combinations:

$$\mathcal{T} \models rx_0...x_n \leftrightarrow \exists \varepsilon \begin{bmatrix} rx_0...x_n \wedge \\ \bigwedge_{i=0}^{n} (p\, x_i \vee \neg p\, x_i) \end{bmatrix},$$

$$\mathcal{T} \models x_0 = x_1 \leftrightarrow \exists \varepsilon \begin{bmatrix} x_0 = x_1 \wedge \\ \bigwedge_{i=0}^{1} (p\, x_i \vee \neg p\, x_i) \end{bmatrix},$$

$$\mathcal{T} \models x_0 = fx_1...x_n \leftrightarrow \exists \varepsilon \begin{bmatrix} x_0 = fx_1...x_n \wedge \\ \bigwedge_{i=0}^{n} (p\, x_i \vee \neg p\, x_i) \end{bmatrix},$$

with $r \in R - \{p\}$, $f \in F$. Thus $\mathcal{T}$ satisfies the first condition of Theorem 3.1.2.

5.3 $\mathcal{T}$ satisfies the second condition

Let us show that every formula φ, without free variables, of the form $\exists \bar{x}'\alpha' \wedge \alpha''$ with $\exists \bar{x}'\alpha' \in A'$ and $\alpha'' \in A''$, is equivalent to *true* or to *false* in $\mathcal{T}$. Since the formula φ does not contain free variables, then there are no reachable variables and no reachable equations in α' from the free variables of $\exists \bar{x}'\alpha'$ and thus according to section 5.1 $\bar{x}' = \varepsilon$. From this and since $\exists \bar{x}'\alpha'$ does not contains free variables we deduce then that $var(\alpha') = \emptyset$. But according to section 5.1 α' is a solved block different from *false*, thus according to the definition of solved blocks (first point) the formula α' is the formula *true*. Then φ is equivalent in $\mathcal{T}$ to the formula without free variables α''. According to section 5.1, the formulas α'' is a solved conjunction of relation blocks and thus according to the definition of the solved blocks (first point) α'' is either the formula *true* or *false* then φ is either equivalent to *false* or to *true* in $\mathcal{T}$. Thus the theory $\mathcal{T}$ satisfies the second condition of Theorem 3.1.2.

5.4 $\mathcal{T}$ satisfies the third condition

Let us first introduce the following convenient definition.

Definition 5.4.1 *Let α be a solved block. We call non-trees formula each block of the form*

$$t_1 = t_2 \wedge \bigwedge_{i=1}^{n} px_i \tag{1}$$

with $\mathrm{var}(t_1 = t_2) = \{x_1, ..., x_n\}$ *and t_1, t_2 M-terms. If* (1) *is a sub-formula of α then we call α-leader of the non-trees formula (1), the α-leader of the M-equation $t_1 = t_2$.*

Let us show that every formula of the form $\exists \bar{x}\, \alpha \wedge \psi$, with $\alpha \in A$ and ψ any formula, is equivalent in $\mathcal{T}$ to a wnfv formula of the form

$$\exists \bar{x}'\, \alpha' \wedge (\exists \bar{x}''\, \alpha'' \wedge (\exists \bar{x}'''\, \alpha''' \wedge \psi))), \tag{2}$$

with $\exists \bar{x}'\alpha' \in A'$, $\alpha'' \in A''$, $\alpha''' \in A$ and $\mathcal{T} \models \forall \bar{x}''\, \alpha'' \rightarrow \exists! \bar{x}'''\alpha'''$.

Let us choose the order $\succ$ such that all the variables of $\bar{x}$ are greater than the free variables of $\exists \bar{x}\alpha$. Let β be the solved formula of α, (β exists according to Property 4.2.8). Let X be the set of the variables of the vector $\bar{x}$. Let Y_{rea} be the set of the reachable variables in β from the free variables of $\exists \bar{x}\beta$ and let Y_{nrea} be the set of the no-reachable variables in β from the free variables of $\exists \bar{x}\beta$. Let us now rename the variables of $Y_{nrea} \cap X$ which have at least an occurrence in a sub-non-trees-formula of β by variables greater than the other variables of β (according to the order $\succ$). Note that theses variables are quantified (they are elements $Y_{nrea} \cap X$), thus we can rename them in $\exists \bar{x}\beta$. Let β^* be the solved formula of β. Let $Lead$ be the set of the β^*-Leaders of all the equations of β^*. If $faux$ is a sub-formula of β^* then: $\bar{x}' = \bar{x}'' = \bar{x}''' = \varepsilon$, $\alpha' = true$, $\alpha'' = false$ and $\alpha''' = true$. Else:

– $\bar{x}'$ contains the variables of $X \cap Y_{rea}$,
– $\bar{x}''$ contains the variables of $(X - Y_{rea}) - Lead$.
– $\bar{x}'''$ contains the variables of $(X - Y_{rea}) \cap Lead$.
– α' is of the form $\alpha'_1 \wedge \alpha'_2$ where α'_1 is the conjunction of (1) all the reachable equations in β^* from the free variables of $\exists \bar{x}\beta^*$, (2) all the sub-non-trees-formulas of β^* whose β^*-leader is not element of $Y_{nrea} \cap X$. α'_2 is the conjunction of all the sub-formulas of β^* of the form $p\, x$ or $\neg p\, x$ with x having at least an occurrence in α'_1.
– α'' is of the form $\alpha''_1 \wedge \alpha''_2$ where α''_1 is the conjunction of all the sub-formulas of β^* of the form $p\, x$ or $\neg p\, x$ with $x \notin \bar{x}'''$, α''_2 is the conjunction of all the sub-formulas of β^* of the form $r t_1 ... t_n$.
– α''' is of the form $\alpha'''_1 \wedge \alpha'''_2$ where α'''_1 is the conjunction of (1) all the equations which are not reachable in β^* from the free variables of $\exists \bar{x}\beta^*$, (2) all the sub-non-trees-formulas of β^* whose β^*-leader belongs to $Y_{nrea} \cap X$. α'''_2 is the conjunction of all the sub-formulas of β^* of the form $p\, x$ or $\neg p\, x$ with x having at least an occurrence in α'''_1.

By construction of the sets A' and A'' it is clear that $\exists \bar{x}'\alpha' \in A'$, $\alpha'' \in A''$ and $\alpha''' \in A$. Moreover, Since $\bar{x}''' = (X - Y_{rea}) \cap Lead$ and since the blocks are solved (thus well typed), then according to the axiom 3 of $\mathcal{T}$ (unique solution) and the definition of solved blocks (point 3 and 4) and the properties of the α-leaders we get $\mathcal{T} \models \forall \bar{x}''\alpha'' \rightarrow \exists! \bar{x}'''\alpha'''$. Thus $\mathcal{T}$ satisfies the third condition of Theorem 3.1.2.

5.5 $\mathcal{T}$ satisfies the fourth condition

Let us show now that $\mathcal{T}$ satisfies the fourth condition of Theorem 3.1.2, i.e. if $\exists \bar{x}'\alpha' \in A'$ then $\mathcal{T} \models \exists ? \bar{x}'\alpha'$. Since $\exists \bar{x}'\alpha' \in A'$ and according to section 5.1, the variables of $\bar{x}'$ are reachable in α' from the free variables of $\exists \bar{x}'\alpha'$. Thus, using Property 4.2.3 we get $\mathcal{T} \models \exists ? \bar{x}'\alpha'$.

Let us show now that if y is a free variable of $\exists\bar{x}'\alpha'$ then $\mathcal{T} \models \exists?y\bar{x}'\,\alpha'$ or there exists $\psi(u) \in \Psi(u)$ such that $\mathcal{T} \models \forall y\,(\exists\bar{x}'\,\alpha') \rightarrow \psi(y)$. Let y be a free variable of $\exists\bar{x}'\alpha'$. Since α' is a solved equation block different from *false* then three cases arise:

Either y occurs in a sub-formula of α' of the form $y = t(\bar{x}', \bar{z}', y) \wedge \neg py$, where $\bar{z}'$ is the set of the free variables of $\exists\bar{x}'\alpha'$ which are different from y, $t(\bar{x}', \bar{z}', y)$ is a term which begins by an element of F followed by variables taken from $\bar{x}'$ or $\bar{z}'$ or $\{y\}$. In this case, the formula $\exists\bar{x}'\alpha'$ implies in $\mathcal{T}$ the formula

$$\exists\bar{x}'\,y = t(\bar{x}', \bar{z}', y) \wedge \neg p\,y,$$

which implies in $\mathcal{T}$ the formula

$$\exists\bar{x}'\bar{z}'w\,y = t(\bar{x}', \bar{z}', w) \wedge \neg p\,y, \tag{3}$$

where $y = t(\bar{x}', \bar{z}', w)$ is the formula $y = t(\bar{x}', \bar{z}', y)$ in which we have replaced every free occurrence of y in the term $t(\bar{x}', \bar{z}', y)$ by the variable w. According to section 5.1, the formula (3) belongs to $\Psi(y)$.

Or y occurs in a sub-formula of α' of the form $y = z \wedge \neg py$. In this case:

1. Recall that according to section 5.1, $\bar{x}'$ contains the quantified reachable variables and the order $\succ$ is such that all the variables of $\bar{x}'$ are greater than the free variables of $\exists\bar{x}'\alpha'$ thus greater than y.
2. According to the definition of solved blocks and α'-leader, we have $y \succ z$ and y is the α'-leader of $y = z$.

From (1) and (2), we deduce that z is a free variable in $\exists\bar{x}'\alpha'$. Since α' is a solved block, y is not α'-leader in any other equation of α' (because all the α'-leaders are distinct), thus y can not appear in another left hand side of an equation of α' (because $\neg px$ and α' is well typed), thus since the variables of $\bar{x}$ are reachable in α' from the free variables of $\exists\bar{x}'\alpha'$ (section 5.1), all the variables of $\bar{x}'$ keep reachable in α' from the free variables of $\exists\bar{x}'y\,\alpha'$ (see Definition4.2.2 of reachable variable). More over, for each value of the free variable z there exists at most a value for y. Using this and Property 4.2.3 we get $\mathcal{T} \models \exists?\bar{x}'y\,\alpha'$.

Or y occurs only in sub-formulas of the form

$$x_0 = t(y) \wedge \neg px_0 \;\; \mathit{or} \;\; t_1 = t_2 \wedge \bigwedge_{i\in I} px_i \tag{4}$$

with $t(y)$ a term which begins by an element of $\mathrm{F} \in F$ and contains at least an occurrence of y. $t_1 = t_2 \wedge \bigwedge_{i\in I} px_i$ is a non-trees-formula. Recall that according to section 5.1, $\bar{x}'$ contains the quantified reachable variables. According to section 5.1, all the variables of $\bar{x}'$ and all the equations of the form $x_0 = t(y) \wedge \neg px_0$ are reachable in α' from the free variables of $\exists\bar{x}'\alpha'$. (1) if $\neg py$ is a sub-formula of α' then since α' is well typed then y does not occur in any non-trees-formula. Since y does not occur in a left-hand side of an equation of α' then the variables of $\bar{x}'y$ keep reachable in α' from the free variables of $\exists\bar{x}'y\,\alpha'$ and thus using Property 4.2.3 we get $\mathcal{T} \models \exists?\bar{x}'y\,\alpha'$. (2) If y occurs in a sub-formula of α' of the form py

then (i) if y occurs in a non-trees-formula of α' then using the last point of the solved block definition we get $\mathcal{T} \models \exists ?y\, \alpha'$, using this and using the fact that α' contains py and not $\neg py$ then the variables of $\bar{x}'$ keep reachable in α' from the free variables of $\exists \bar{x}' y\, \alpha'$ and thus using Property 4.2.3 we get $\mathcal{T} \models \exists ? \bar{x}' y\, \alpha'$ (ii) else since y does not occur in a left-hand side of an equation of α' of the form $y = fx_1...x_n$ or $y = x_1$ then the variables of $\bar{x}' y$ keep reachable in α' from the free variables of $\exists \bar{x}' y\, \alpha'$ and thus using Property 4.2.3 we get $\mathcal{T} \models \exists ? \bar{x}' y\, \alpha'$.

In all the other cases the blocks are not well typed thus not solved and thus can not be element of A'. In all cases $\mathcal{T}$ satisfies the fourth condition of Theorem 3.1.2.

5.6 $\mathcal{T}$ satisfies the fifth condition

$\mathcal{T}$ satisfies the first point of the fifth condition Let us show that if $\alpha'' \in A''$ then the formula $\neg\alpha''$ is equivalent in $\mathcal{T}$ to a disjunction of elements of A, i.e. a disjunction of blocks. Let α'' a formula of A''.

According to section 5.1, either α'' is the formula *false* and thus $\neg\alpha''$ is the formula *true* which clearly belongs to A'', or α'' is a formula of the form

$$\beta \wedge (\bigwedge_{x \in X} px) \wedge (\bigwedge_{y \in Y} \neg py),$$

with β a conjunction of M-relations β_k with $var(\beta) \subseteq X \cup Y$. According to the second point of the definition of the flexible theory we have $\mathcal{T} \models \neg\beta \leftrightarrow \beta'$ where β' is a disjunction of M-relations and M-equations. Thus, the formula $\neg\alpha''$ is equivalent in $\mathcal{T}$ to a wnfv formula of the form

$$(\bigvee_{k \in K} (\beta'_k \wedge p_k)) \vee (\bigvee_{x \in X} \neg px) \vee (\bigvee_{y \in Y} py),$$

where β'_k is an M-equation or an M-relation, p_k is a conjunction of formula of the form px for all variable $x \in var(beta'_k)$. This formula is clearly a disjunction of blocks. In all cases $\mathcal{T}$ satisfies the first point the fifth condition of Theorem 3.1.2.

$\mathcal{T}$ satisfies the second point of the fifth condition Let us show that if $\alpha'' \in A''$ then for every variable x''_ℓ, the formula $\exists x''_\ell \alpha''$ is equivalent in $\mathcal{T}$ to an element of A''. Let α'' be a formula of A'', three cases arise:

If x''_ℓ has no occurrences in α'' then the formula $\exists x''_\ell\, \alpha''$ is equivalent in $\mathcal{T}$ to α'' which belongs to A''.

If the formula $\exists x''_\ell\, \alpha''$ is of the form $\exists x''_\ell\, \alpha''_1 \wedge \neg p\, x''_\ell$ with $\alpha''_1 \in A''$ and x''_ℓ has no occurrences in α''_1 then the formula $\exists x''_\ell\, \alpha''$ is equivalent in $\mathcal{T}$ to $\alpha''_1 \wedge (\exists x''_\ell\, \neg p\, x''_\ell)$, which, according to our axiomatization is equivalent in $\mathcal{T}$ to α''_1, which belongs to A''.

If the formula $\exists x''_\ell\, \alpha''$ is of the form

$$\exists x''_\ell\, \alpha''_1 \wedge \varphi$$

with φ a relation block containing at least a formula of the form $rt_1...t_n$ with $r \in R_M$ and x''_ℓ has no occurrences in α''_1, then the formula $\exists x''_\ell\, \alpha''$ is equivalent in $\mathcal{T}$ to

$$\alpha''_2 \wedge (\exists x''_\ell\, \varphi),$$

with α''_2 a block of the form $\alpha''_1 \wedge \phi$ where ϕ is a conjunction of sub-formulas of α''_1 of the form $\neg px$ or px with $x \in \mathit{var}((\alpha''_1 \cap \varphi) - \{x''_\ell\})$. According to the last point of the definition of flexible theory, the preceding formula is wnfv equivalent in $\mathcal{T}$ to false or to to

$$\alpha''_2 \wedge \varphi''.$$

Since α''_2 belongs to A'' and φ does not contains new variables (wnfv) then preceding formula belongs to A''. In all cases $\mathcal{T}$ satisfies the second point of the fourth condition of Theorem 3.1.2. □

$\mathcal{T}$ satisfies the third point of the fifth condition First, we present two properties which hold in any model $\mathcal{M}$ of $\mathcal{T}$. The first one results from the axiomatization of $\mathcal{T}$ and introduce the notion of *zero-infinite* in $\mathcal{M}$. The second one is due to the fact that T is flexible thus all model M of T is also flexible (more exactly the last point of the definition of flexible model).

Property 5.6.1 *Let F_0 the set of the 0-ary function symbols of F. Let $\mathcal{M}$ be a model of $\mathcal{T}$ and let $f \in F - F_0$. The set of the individuals i of $\mathcal{M}$, such that $\mathcal{M} \models \neg p\, i$ and the set of the individuals i of $\mathcal{M}$, such that $\mathcal{M} \models \exists \bar{x}\, i = f\bar{x} \wedge \neg pi$, are infinite.*

Property 5.6.2 *Let $\mathcal{M}$ be a model of $\mathcal{T}$. Let $\bigwedge_{j\in J} r_j(x)$ be a conjunction of relation blocks. Let $\exists x \bigwedge_{j\in J} r'_j(x)$ be an instantiation of $\exists x \bigwedge_{j\in J} r_j(x)$ by individuals of $\mathcal{M}$. Let $\varphi(x)$ the formula*

$$p\, x \wedge \bigwedge_{j\in J} r'_j(x). \tag{5}$$

The set of the individuals i of $\mathcal{M}$ such that $\mathcal{M} \models \varphi(i)$ is empty or infinite.

Let $\mathcal{M}$ be a model of $\mathcal{T}$. Recall that $\Psi(u)$ is the set of formulae of the form $\exists \bar{y}\, u = f\bar{y} \wedge \neg p\, u$, with $f \in F - F_0$. Let $\varphi(x)$ be a formula which belongs to A'', let us show that, for every variables x we have $\mathcal{T} \models \exists_{o\,\infty}^{\Psi(u)} x\, \varphi(x)$. Let $\mathcal{M}$ be a model of $\mathcal{T}$ and let $\exists x\, \varphi'(x)$ be an any instantiation of $\exists \bar{x}\, \varphi(x)$ by individuals of M such that $\mathcal{M} \models \exists x\, \varphi'(x)$. Having an any condition of the form

$$\mathcal{M} \models \varphi'(i) \wedge \neg\psi_1(i) \wedge \cdots \wedge \neg\psi_n(i),$$

with $\psi_j(u) \in \Psi(u)$, it is enough to show that there exists an infinity of individuals i of $\mathcal{M}$ which satisfy this condition. This condition can be replaced by the following stronger condition

$$\mathcal{M} \models \begin{pmatrix} p\, i \vee \\ \psi_{n+1}(i) \end{pmatrix} \wedge \varphi'(i) \wedge \neg\psi_1(i) \wedge \cdot\cdot \wedge \neg\psi_n(i),$$

where $\psi_{n+1}(u)$ is an element of $\Psi(u)$ which has been chosen different from $\psi_1(u), \ldots, \psi_n(u)$, (always possible because the set $F - F_M$ is infinite). Since for every k between 1 and n, we have:

- $\mathcal{T} \models p\, x \rightarrow \neg\psi_k(x)$
- $\mathcal{T} \models \psi_{n+1}(x) \rightarrow \neg\psi_k(x)$ (axiom 2of $\mathcal{T}$ conflict of symbol).

The preceding condition is simplified to

$$\mathcal{M} \models (p\, i \wedge \varphi'(i)) \vee (\psi_{n+1}(i) \wedge \varphi'(i))$$

and thus, knowing that $\mathcal{M} \models \exists x\, \varphi'(x)$, it is enough to show that there exists an infinity of individuals i of $\mathcal{M}$ such that

$$\mathcal{M} \models p\, i \wedge \varphi'(i) \;\text{ or }\; \mathcal{M} \models \psi_{n+1}(i) \wedge \varphi'(i). \tag{6}$$

Two cases arise:

Either the formula $p\, x$ occurs in $\varphi'(x)$. Since $\varphi'(x)$ is an instantiation of a solved relation block and $M \models \exists x\, \varphi'(x)$, the formula $p\, x \wedge \varphi'(x)$ is equivalent in $\mathcal{M}$ to a M-formula of the form (5). According to Property 5.6.2 and since $\mathcal{M} \models \exists x\, p\, x \wedge \varphi'(x)$, there exists an infinity of individuals i of $\mathcal{M}$ such that $\mathcal{M} \models p\, i \wedge \varphi'(i)$ and thus, such that (6).

Or, the formula $p\, x$ does not occur in $\varphi'(x)$. Since $\varphi'(x)$ is an instantiation of a solved relation block and $\mathcal{M} \models \exists x\, \varphi'(x)$, the M-formula $\psi_{n+1}(x) \wedge \varphi'(x)$ is equivalent in $\mathcal{M}$ to $\psi_{n+1}(x)$. According to Property 5.6.1 there exists an infinity of individuals i of $\mathcal{M}$ such that $\mathcal{M} \models \psi_{n+1}(i)$, thus, such that $\mathcal{M} \models \psi_{n+1}(i) \wedge \varphi'(i)$ and thus such that (6). □

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